The Automorphism Group of a Structural Matrix Algebra

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ABSTRACT

We characterize the automorphism group of certain subalgebras of matrix algebras with entries from a field K, known as structural matrix algebras. These include the algebras of upper triangular matrices. We also give necessary and sufficient conditions for every K-automorphism of such a subalgebra to be inner.

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1. INTRODUCTION

Automorphisms of certain subalgebras of matrix algebras have been the object of several recent papers. Jøndrup [6] has shown that if A is a simple algebra, finite dimensional over its center K, then all K-automorphisms of the algebra of upper triangular matrices over A are inner. Similarly, Barker and Kezlan [3] have proved that every R-automorphism of the algebra of upper triangular matrices with entries from an integral domain R is inner. Related results appeared in [1], [2], [5] and [7].

In this paper we characterize the group of K-automorphisms of certain subalgebras of a matrix algebra with entries from a field K. Also, we give necessary and sufficient conditions for every K-automorphism of such a

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subalgebra to be inner. In this respect, our results include those from [3] and [6], when the ring of coefficients is a field (see Corollary to Theorem D).

We would like to remark that the arguments of Section 3 actually hold when entries are taken on a simple algebra that is finite dimensional over its center (with very slight changes). Therefore, Theorems A and B remain valid when the ring of coefficients is such an algebra.

2. NOTATION AND MAIN RESULTS

Let $M_n(K)$ be the ring of $n \times n$ matrices over a field K. The unity element of $M_n(K)$ is denoted by I_n. Given $A \in M_n(K)$, A_{ij} denotes the (i,j) entry of A. The matrix unit having 1 in the $(i,j)^{\text{th}}$ position and zeros elsewhere is denoted by E^{ij}. The matrix E^{ii} is written simply E^i.

If ρ is a relation on $I = \{1, 2, \dots, n\}$ which is reflexive and transitive, then the set

$$S = S(\rho, K) = \left\{A \in M_n(K) \,\middle|\, A_{ij} = 0 \text{ if } (i,j) \notin \rho\right\}$$

is a subalgebra of $M_n(K)$ (see [10]). Following [10], we call $S(\rho, K)$ the *structural matrix algebra over K defined by the relation* ρ.

The group of K-automorphisms of S is denoted by Aut S. In what follows, the word automorphism means K-automorphism.

For each invertible matrix $A \in S$, we denote by C_A the inner automorphism induced by A. Then the set

$$\mathscr{C} = \{C_A \mid A \in S \text{ is invertible}\}$$

is a normal subgroup of Aut S.

A permutation σ of I is said to be an *automorphism of* ρ if $(\sigma(i), \sigma(j)) \in \rho$ for all $(i,j) \in \rho$. Such a permutation gives rise to an automorphism $\hat{\sigma}$ of S on defining

$$\hat{\sigma}(E^{ij}) = E^{\sigma(i)\sigma(j)}, \qquad (i,j) \in \rho,$$

and extending linearly. The set of automorphims $\hat{\sigma}$ of S such that $\sigma(i) < \sigma(j)$ whenever $(i,j), (j,i) \in \rho$ and $i < j$ is a subgroup of Aut S, which we shall denote by $\mathscr{P}$.

In the case when S is semisimple, these groups allow us to describe Aut S.

THEOREM A. *Let S be a semisimple matrix algebra. Then*

$$\mathrm{Aut}\,(S) = \mathscr{C} \rtimes \mathscr{P},$$

the semidirect product of $\mathscr{C}$ by $\mathscr{P}$.

When S is a semisimple algebra, it follows that ρ is an equivalence relation and we have:

THEOREM B. *Let S be a semisimple structural matrix algebra. Then every K-automorphism of S is inner if and only if all the equivalence classes of ρ are of different sizes.*

Following [8], we say that a function $g : \rho \to K^*$ is *transitive* if

$$g(i,j)g(j,k) = g(i,k)$$

for all $(i,j),(j,k) \in \rho$. Such a function is said to be *trivial* if there exists a function $s : I \to K^*$ such that $g(i,j) = s(i)s(j)^{-1}$ for all $(i,j) \in \rho$.

Every transitive function $g : \rho \to K^*$ gives rise to an automorphism g^* of S on defining

$$g^*(E^{ij}) = g(i,j)E^{ij}, \qquad (i,j) \in \rho,$$

and extending linearly.

Now, let $\bar{\rho}$ be the following relation on I:

$$(i,j) \in \bar{\rho} \quad \text{if and only if} \quad (i,j),(j,i) \in \rho,$$

and let Δ be the graph associated to the relation $\rho \setminus \bar{\rho}$ [that is, the vertices of Δ are the elements i of I such that either (i,j) or $(j,i) \in \rho \setminus \bar{\rho}$ for some $j \in I$, and the edges of Δ are the unordered pairs $\{i,j\}$ such that either (i,j) or $(j,i) \in \rho \setminus \bar{\rho}$]. For each connected component Δ_l of Δ, we consider a tree $T_l \subset \Delta_l$ containing every vertex of Δ_l [see 4, §2, Corollary 5]. We fix one such tree T_l for each index l.

Let $\bar{\bar{\rho}}$ be the subset of ρ such that

$$(i,j) \in \bar{\bar{\rho}} \quad \text{if and only if} \quad (i,j) \in \rho \text{ and } \{i,j\} \text{ is an edge of } \textstyle\bigcup_l T_l.$$

Furthermore, let V be the set of vertices of $\bigcup T_l$. Let also $J = I \setminus V$, and set $\bar{\bar{\bar{\rho}}} = \bar{\rho} \cap J \times J$. Then, the set

$$\mathscr{G} = \left\{ g^* \in Aut\, S \,\middle|\, g : \rho \to K^* \text{ and } g(i,j) = 1 \text{ for all } (i,j) \in \bar{\bar{\rho}} \cup \bar{\bar{\bar{\rho}}} \right\}$$

is a subgroup of Aut S.

THEOREM C. *Let S be a structural matrix algebra. Then*

$$\operatorname{Aut} S = (\mathscr{C} \rtimes \mathscr{G}) \rtimes \mathscr{P}.$$

THEOREM D. *Let S be a structural matrix algebra. Then every K-automorphism of S is inner if and only if the following conditions hold*:

(i) *every transitive mapping $g : \rho \to K^*$ is trivial*;
(ii) *every automorphism of ρ fixes the equivalence classes of $\bar{\rho}$.*

3. THE SEMISIMPLE CASE

We begin with some lemmas.

LEMMA 3.1. *Let $e \neq 0$ be a central idempotent of S. Then there exists a subset J of I such that*

$$e = \sum_{j \in J} E^j.$$

Proof. For all $i \in I$, there exists $x_i \in K$ such that $E^i e E^i = x_i E^i$. Since $E^i e E^i$ is an idempotent, we must have that either $x_i = 0$ or $x_i = 1$. Writing

$$e = I_n e I_n = \sum_{i \in I} E^i e E^i,$$

the conclusion follows. ■

LEMMA 3.2. *The set*

$$\mathscr{I} = \left\{A \in S \middle| A_{ij} = 0 \text{ if } (i,j) \in \bar{\rho}\right\}$$

is an ideal of S.

Proof. Take $B \in S$ and $A \in \mathscr{I}$, and set $C = BA$. Suppose now that $(i,j) \in \bar{\rho}$. Then

$$C_{ij} = \sum_{k \in I} B_{ik} A_{kj}.$$

If, for some k, we had that both $B_{ik} \neq 0$ and $A_{kj} \neq 0$, then we would have that $(j,i), (i,k) \in \rho$. Hence, we can conclude that $(j,k) \in \rho$. But, since $A_{kj} \neq 0$, we must have that $(k, j) \notin \rho$, a contradiction. Then $C_{ij} = 0$ and $C \in \mathscr{J}$.

A similar argument shows that $AB \in \mathscr{J}$. ∎

LEMMA 3.3. *If S is a semisimple algebra, then $\mathscr{J} = 0$.*

Proof. In fact, by Lemma 3.2, we have that $\mathscr{J}$ is an ideal, hence a direct summand of S, i.e.,

$$S = \mathscr{J} \oplus \mathscr{J}',$$

where $\mathscr{J}'$ is an ideal of S.

Let e be the unity element of $\mathscr{J}$. We have that either

$$e = \sum_{j \in J} E^j \quad \text{or} \quad e = 0,$$

due to Lemma (3.1). But $\mathscr{J}$ contains no diagonal matrices, hence $e = 0$. ∎

We are now ready to give a characterization of a semisimple structural matrix algebra. This result was proved in [9] following a different approach.

THEOREM 3.4. *Let $S = S(\rho, K)$ be a structural matrix algebra. Then S is semisimple if and only if ρ is symmetric.*

Proof. Suppose that S is semisimple. By Lemma 3.3, we must have that $\mathscr{J} = 0$; thus, ρ is symmetric.

For the converse, let $L \neq 0$ be an ideal of S. We shall show that L is a direct summand of S.

Given $A \in L$, with $A_{kl} \neq 0$, we have that

$$E^k A E^l = A_{kl} E^{kl} \in L.$$

Hence, $E^{kl} \in L$. Since ρ is symmetric, we conclude that $E^{lk} \in S$. Therefore, $E^l = E^{lk} E^{kl} \in L$.

Now, let $T = \{t \in I \mid E^t \in L\}$ and set $e = \sum_{t \in T} E^t$. We claim that $L = Se$.

Clearly, we have that $Se \subset L$. Now, if $A \in L$ with $A_{kl} \neq 0$, we have that $A_{kl} E^{kl} \in L$ as we have shown above.

Then, in order to prove that $L \subset Se$, it is enough to show that $E^{kl} \in Se$. But this follows immediately from the equation $E^{kl} = E^{kl} e$.

Now, if $E^{kl} \in L$, we have that $l \in T$, as before. A similar argument shows that also $k \in T$.

Hence, it is easy to conclude that e is a central idempotent of S, and the claim follows. ■

Before proving Theorem A, we need to characterize the simple components of S.

LEMMA 3.5. *Let S be semisimple, and let $e \in S$ be an idempotent such that Se is a simple component of S. Then there exists one and only one equivalence class C of ρ such that*

$$e = \sum_{x \in C} E^x.$$

Proof. By Lemma 3.1, we have that $e = \sum_{j \in J} E^j$ for a suitable subset J of I.

Pick $j \in J$ and $(k, j) \in \rho$. We have that

$$E^{kj} e = E^{kj}.$$

Then we must have $eE^{kj} = E^{kj}$, and we conclude that $k \in J$.

We have shown that, if $j \in J$, the equivalence class of j is contained in J. Then J is a union of equivalence classes. But, if there were two or more such classes, it is easy to see that e would be decomposable. ■

LEMMA 3.6. *With the notation of Lemma 3.5, we have that*

$$Se = \sum_{x_k, x_l \in C} KE^{x_k x_l},$$

and consequently, $\dim_K Se = |C|^2$ *(where $|C|$ denotes the order of C).*

Proof. If $x_k, x_l \in C$, we have that $(x_k, x_l) \in \rho$. Then $E^{x_k x_l} = E^{x_k x_l} e \in Se$, due to the expression for e given by Lemma 3.5. We have shown that $\sum_{x_k, x_l} KE^{x_K x_l} \subset Se$.

For the other inclusion, we observe that if $A \in Se$ then $A = Ae = eA$. From this, it is easy to conclude that A is of the required form. ■

Proof of Theorem A. Let $S = S_1 \oplus S_2 \oplus \cdots \oplus S_r$, where the S_i are the simple components of S, and let ϕ be an automorphism of S. We have that

$$\phi(S_i) = S_{f(i)}, \qquad 1 \leqslant i \leqslant r,$$

where f is a bijection of $\{1, 2, \ldots, r\}$. Furthermore, if C_i denotes the equivalence class of ρ corresponding to the unity element of S_i given by Lemma 3.5, we have that $|C_i| = |C_{f(i)}|$, in view of Lemma 3.6.

We now enumerate explicitly $C_i = \{x_1, x_2, \ldots\}$ and $C_{f(i)} = \{y_1, y_2, \ldots\}$, where $x_k < x_l$ and $y_k < y_l$ if $k < l$. We can define a bijection from C_i to $C_{f(i)}$, mapping x_k to the corresponding y_k. As this can be done in each equivalence class of ρ, this process defines a permutation of I, which we shall denote by σ. It is easy to see that σ is an automorphism of ρ; hence, it induces an automorphism $\hat{\sigma}$ of S.

We now fix a simple component S_i of S. As we saw above, we have that $\phi(S_i) = S_{f(i)}$. Given an element $X \in S_{f(i)}$, we have that

$$(\hat{\sigma})^{-1}(X) \in S_i,$$

since

$$(\hat{\sigma})^{-1}(E^{y_k y_l}) = E^{x_k x_l} \in S_i$$

for all $E^{y_k y_l} \in S_{f(i)}$ (because of Lemma 3.6). Then the map

$$X \in S_{f(i)} \overset{(\hat{\sigma})^{-1}}{\mapsto} (\hat{\sigma})^{-1}(X) \in S_i \overset{\phi}{\mapsto} \left(\phi \circ (\hat{\sigma})^{-1}\right)(X) \in S_{f(i)}$$

is an automorphism of $S_{f(i)}$. But $S_{f(i)}$ is a simple algebra finite dimensional over its center. By the Skolem-Noether theorem, there exists an invertible matrix $A_{f(i)} \in S_{f(i)}$ such that this automorphism is conjugation by $A_{f(i)}$.

Set now $A = A_{f(1)} + \cdots + A_{f(r)}$. We claim that $\phi \circ (\hat{\sigma})^{-1} = C_A$. In fact, it is enough to prove the equality in each component $S_{f(i)}$. Given a matrix unit $E^{y_k y_l} \in S_{f(i)}$, we have that

$$C_A(E^{y_k y_l}) = AE^{y_k y_l}A^{-1} = A_{f(i)}E^{y_k y_l}A_{f(i)}^{-1} = \left(\phi \circ (\hat{\sigma})^{-1}\right)(E^{y_k y_l}),$$

and the claim is proved. Therefore, $\phi = C_A \circ \hat{\sigma}$.

To conclude the proof, we notice first that $\hat{\sigma} \in \mathscr{P}$. Also, $\mathscr{C} \cap \mathscr{P} = 1$. In fact, given $\hat{\sigma} \in \mathscr{C} \cap \mathscr{P}$, we have that $\hat{\sigma}$ is a conjugation by the matrix $A \in S$. Therefore, $\hat{\sigma}(S_i) = S_i$ for all i. Then $\sigma(C_i) = C_i$ for all i, and by the definition of σ it is easy to conclude that $\sigma = 1$.

Since $\mathscr{C} \lhd \operatorname{Aut} S$, the proof is finished. ∎

A little extra effort gives Theorem B.

Proof of Theorem B. First, we observe that if σ is an automorphism of ρ and C is an equivalence class of ρ, then $\sigma(C)$ is also an equivalence class of ρ.

Suppose first that the equivalence classes of ρ are of different sizes. In view of Theorem A, it is enough to prove that $\mathscr{P} = 1$. Set $\hat{\sigma} \in \mathscr{P}$. By the hypothesis and the above, we have that $\sigma(C) = C$ for every equivalence class C of ρ.

Now, let $C = \{x_1, x_2, \dots\}$, where $x_k < x_l$ if $k < l$. As $\hat{\sigma} \in \mathscr{P}$, we have that $\sigma(x_k) < \sigma(x_l)$. Thus, it is easy to see that $\sigma(x_k) = x_k$ for all x_k, and hence $\hat{\sigma} = 1$.

For the converse, we observe that $\mathscr{P} = 1$ by hypothesis. Suppose there are two equivalence classes C_1 and C_2 of ρ such that $|C_1| = |C_2|$, and write

$$C_1 = \{x_1, x_2, \dots\}, \qquad C_2 = \{y_1, y_2, \dots\},$$

where $x_k < x_l$ and $y_k < y_l$ if $k < l$. Then we can define the following permutation σ of I:

$$\begin{aligned} \sigma(x_k) &= y_k, & \forall x_k &\in C_1, \\ \sigma(y_l) &= x_l, & \forall y_l &\in C_2, \\ \sigma(z) &= z, & \forall z &\notin C_1 \cup C_2. \end{aligned}$$

Then, $\hat{\sigma} \in \mathscr{P}$, a contradiction. ■

4. THE GENERAL CASE

First, we need to describe $\mathscr{J}(S)$, the Jacobson radical of S. In what follows, we shall denote $\mathscr{J}(S)$ simply by $\mathscr{J}$.

The following proposition can be obtained as a consequence of the results in [9]. We offer here a direct proof.

PROPOSITION 4.1. *With the notation above, we have that*

$$\mathscr{J} = \left\{ A \in S \,\middle|\, A_{ij} = 0 \text{ if } (i,j) \in \vec{\rho} \right\}.$$

Proof. Let

$$\mathscr{I} = \left\{ A \in S \,|\, A_{ij} = 0 \text{ if } (i,j) \in \vec{\rho} \right\}.$$

By Lemma 3.2, $\mathcal{I}$ is an ideal of S. We claim that

$$\frac{S}{\mathcal{I}} = \left\{A + \mathcal{I} \,\middle|\, A_{ij} = 0 \text{ if } (i,j) \notin \bar{\rho}\right\}.$$

In fact, if $B + \mathcal{I} \in \frac{S}{\mathcal{I}}$, choose $B' \in S$ such that $B'_{ij} = B_{ij}$ if $(i,j) \in \bar{\rho}$, and $B'_{ij} = 0$ if $(i,j) \notin \bar{\rho}$. Then $B - B' \in \mathcal{I}$ and $B + \mathcal{I} = B' + \mathcal{I} \in \{A + \mathcal{I} | A_{ij} = 0 \text{ if } (i,j) \notin \bar{\rho}\}$. The opposite inclusion is trivial.

Now, let

$$S' = \left\{A \in M_n(K) \,\middle|\, A_{ij} = 0 \text{ if } (i,j) \in \bar{\rho}\right\}.$$

Then the mapping $f : S' \to \frac{S}{\mathcal{I}}$ given by $f(A) = A + \mathcal{I}$ $\forall A \in S'$ is an isomorphism of algebras. By Theorem 3.4, S' is semisimple, since it is a structural matrix algebra defined by the symmetric relation $\bar{\rho}$. Then $\frac{S}{\mathcal{I}}$ is semisimple; hence $\mathcal{J} \subset \mathcal{I}$.

Now, $\frac{\mathcal{I}}{\mathcal{J}}$ is a semisimple component of $\frac{S}{\mathcal{J}}$. Let $e + \mathcal{J}$ be its unity element, where $e \in \mathcal{I}$. We compute

$$e + \mathcal{J} = (I_n + \mathcal{J})(e + \mathcal{J})(I_n + \mathcal{J}) = \left(\sum_{i \in I} E^i + \mathcal{J}\right)(e + \mathcal{J})\left(\sum_{i \in I} E^i + \mathcal{J}\right)$$

$$= \sum_{i \in I} E^i e E^i + \mathcal{J}.$$

That is,

$$e - \sum_{i \in I} E^i e E^i = \eta \in \mathcal{J}.$$

Since $\mathcal{J} \subset \mathcal{I}$, we have that $\sum E^i e E^i = e - \eta \in \mathcal{I}$. But $\mathcal{I}$ does not contain diagonal matrices. Hence, $e - \eta = 0$ and $e = \eta \in \mathcal{J}$. That is, $\mathcal{I} \subset \mathcal{J}$. ■

In order to deal with the automorphisms of S, we observe that given $\phi \in \operatorname{Aut} S$ we have that $\phi(\mathcal{J}) = \mathcal{J}$. So ϕ induces an automorphism $\bar{\phi}$ of $\frac{S}{\mathcal{J}}$, namely,

$$\bar{\phi}(A + \mathcal{J}) = \phi(A) + \mathcal{J} \qquad \text{for all} \quad A \in S.$$

Suppose now that e, e' are idempotents of S such that $\bar{\phi}(e + \mathcal{J}) = e' + \mathcal{J}$. Since $\bar{\phi}(e + \mathcal{J}) = \phi(e) + \mathcal{J}$, we have that $\phi(e)$ must be of the form

$\phi(e) = e' + \eta$, where $\eta \in \mathscr{J}$. We need more information about these idempotents.

LEMMA 4.2. *Let $E = E^j$, $\eta \in \mathscr{J}$, and let $\mathscr{E} = E + \eta$ be an idempotent of S. Then for all $i \geqslant 3$ we have that*

$$\mathscr{E} = E + (\eta + \eta^2 + \cdots + \eta^{i-1})E + E(\eta + \eta^2 + \cdots + \eta^{i-1}) + \sum_{k=1}^{i-2} (\eta + \eta^2 + \cdots + \eta^{i-1-k})E\eta^k + \eta^i.$$

Proof. We shall proceed by induction on i. We have that

$$\mathscr{E} = \mathscr{E}^2 = E^2 + E\eta + \eta E + \eta^2 = E + E\eta + \eta E + \eta^2.$$

Multiplying the first and last members of the equation by $\mathscr{E}$ $(= E + \eta)$, we obtain

$$\mathscr{E} = E + E\eta E + \eta E + \eta^2 E + E\eta + E\eta^2 + \eta E\eta + \eta^3.$$

But $E\eta E = \eta_{jj}E^j$, $\eta_{jj} \in K$. As $E\eta E \in \mathscr{J}$ and $\mathscr{J}$ does not contain diagonal matrices, we conclude that $E\eta E = 0$. So, $\mathscr{E} = E + (\eta + \eta^2)E + E(\eta + \eta^2) + \eta E\eta + \eta^3$, and the claim is proved for $i = 3$.

Suppose now that equality holds for a given index i, and multiply both sides of the equality by $\mathscr{E}(= E + \eta)$. Observing that the terms $E(\eta + \eta^2 + \cdots + \eta^{i-1})E$ and $\Sigma_k(\eta + \eta^2 + \cdots + \eta^{i-k-1})E\eta^k E$ are both zero (by the argument used above), we get

$$\mathscr{E} = E + (\eta + \eta^2 + \cdots + \eta^i)E + E(\eta + \eta^2 + \cdots + \eta^i) + (\eta + \eta^2 + \cdots + \eta^{i-1})E\eta + \sum_{k=1}^{i-2} (\eta + \eta^2 + \cdots + \eta^{i-1-k})E\eta^{k+1} + \eta^{i+1},$$

as required. ■

Notice that, since S is artinian, the elements of $\mathscr{J}$ are nilpontent.

COROLLARY 4.3. *With the notation of Lemma 4.2, we have that:*

(i) *if the index of nilpotency of η is 2, then $\mathscr{E} = E + \eta E + E\eta$;*
(ii) *if the index of milpotency of η is $s > 2$, then*

$$\mathscr{E} = E + (\eta + \eta^2 + \cdots + \eta^{s-1})E + E(\eta + \eta^2 + \cdots + \eta^{s-1}) + \sum_{k=1}^{s-2} (\eta + \eta^2 + \cdots + \eta^{s-1-k})E\eta^k.$$

Proof. Since $\mathscr{E} = E + E\eta + \eta E + \eta^2$, (i) follows trivially
In a similar way, (ii) follows from the lemma above. ■

COROLLARY 4.4. *Let $\mathscr{E}$ be an idempotent of S under the conditions of Lemma 4.2. Then there exists $\theta \in \mathscr{J}$ such that*

$$\mathscr{E} = E + E\theta + \theta E + \theta E \theta.$$

Conversely, if $\theta \in \mathscr{J}$ and $E = E^j$ $(j \in I)$, then

$$\mathscr{E} = E + E\theta + \theta E + \theta E \theta$$

is an idempotent of S.

Proof. Suppose $\mathscr{E}$ is an idempotent as in Lemma 4.2. If the index of nilpotency of η is 2, then $\mathscr{E} = E + E\eta + \eta E$, by Corollary 4.3. Computing $\mathscr{E}^2$, we get that

$$E + E\eta + \eta E = \mathscr{E} = \mathscr{E}^2 = E + E\eta + \eta E + \eta E \eta.$$

Hence, $\eta E \eta = 0$, and it is enough to choose $\theta = \eta$. If the index of nilpotency is $s > 2$, computing $\mathscr{E}^2$, we obtain

$$\mathscr{E} = \mathscr{E}^2 = E + (\eta + \eta^2 + \cdots + \eta^{s-1})E + E(\eta + \eta^2 + \cdots + \eta^{s-1})$$
$$+ (\eta + \eta^2 + \cdots + \eta^{s-1})E(\eta + \eta^2 + \cdots + \eta^{s-1}).$$

Then it is enough to choose $\theta = \eta + \eta^2 + \cdots + \eta^{s-1}$.
The converse is trivial. ■

LEMMA 4.5. *Let $\theta \in \mathscr{J}$, $E = E^j$ $(j \in I)$, and let $\mathscr{E}$ be the idempotent*

$$\mathscr{E} = E + E\theta + \theta E + \theta E \theta.$$

Then

$$\mathscr{U} = (I_n + E\theta)(I_n - \theta E) \in S$$

is invertible, and

$$\mathscr{U}\mathscr{E}\mathscr{U}^{-1} = E.$$

Proof. We have that

$$(I_n - \theta E)(I_n + \theta E) = I_n,$$
$$(I_n + E\theta)(I_n - E\theta) = I_n.$$

Therefore, $\mathscr{U}^{-1} = (I_n + \theta E)(I_n - E\theta)$. An easy calculation shows that

$$(I_n - \theta E)\mathscr{E}(I_n + \theta E) = E + E\theta,$$
$$(I_n + E\theta)(E + E\theta)(I_n - E\theta) = (I_n + E\theta)E(I_n + E\theta)(I_n - E\theta)$$
$$= (I_n + E\theta)E = E. \qquad \blacksquare$$

Now, for each $j \in I$, pick $\theta_j \in \mathscr{J}$ and consider

$$\mathscr{E}_j = E^j + E^j\theta_j + \theta_j E^j + \theta_j E^j \theta_j,$$
$$\mathscr{U}_j = \left(I_n + E^j\theta_j\right)\left(I_n - \theta_j E_j\right).$$

LEMMA 4.6. *With the notation above, we have that:*

(i) $\overline{C_{\mathscr{U}_j}} = 1$ *for all* $j \in I$.
(ii) *If* $E^i\mathscr{E}_j = \mathscr{E}_j E^i = 0$, *where* $i, j \in I$, $i \neq j$, *then* $E^i\mathscr{U}_j = \mathscr{U}_j E^i = E^i$.

Proof. Since the elements $\mathscr{U}_j$ are such that $\mathscr{U}_j + \mathscr{J} = I_n + \mathscr{J}$ for all $j \in I$, (i) follows.

In order to prove (ii), we compute

$$E^i\mathscr{E}_j = E^i\theta_j E^j + E^i\theta_j E^j\theta_j = E^i\theta_j E^j\left(I_n + \theta_j\right) = 0,$$
$$\mathscr{E}_j E^i = E^j\theta_j E^i + \theta_j E^j\theta_j E^i = \left(I_n + \theta_j\right)\left(E^j\theta_j E^i\right) = 0$$

Since $\theta_j \in \mathscr{J}$, we have that $I_n + \theta_j$ is invertible. Hence, $E^i\theta_j E^j = E^j\theta_j E^i = 0$. Now, we compute $E^i\mathscr{U}_j$ and $\mathscr{U}_j E^i$:

$$E^i\mathscr{U}_j = E^i\left(I_n + E^j\theta_j\right)\left(I_n - \theta_j E^j\right) = E^i\left(I_n - \theta_j E^j\right) = E^i - E^i\theta_j E^j = E^i,$$
$$\mathscr{U}_j E^i = \left(I_n + E^j\theta_j\right)\left(I_n - \theta_j E^j\right)E^i = \left(I_n + E^j\theta_j\right)E^i = E^i + E^j\theta_j E^i = E^i.$$

$\blacksquare$

LEMMA 4.7. *Let φ be an automorphism of S such that there exists a permutation σ of I satisfying*

$$\overline{\varphi}(E^j + \mathscr{J}) = E^{\sigma(j)} + \mathscr{J} \qquad \text{for all} \quad j \in I.$$

Then there exists an invertible element $\mathscr{U}$ of S such that $\overline{C_{\mathscr{U}}} = 1$ and

$$(C_{\mathscr{U}} \circ \varphi)(E^j) = E^{\sigma(j)} \qquad \text{for all} \quad j \in I.$$

Proof. We have that

$$\varphi(E^1) = E^{\sigma(1)} + \eta_1, \quad \text{where} \quad \eta_1 \in \mathscr{J}.$$

Then $E^{\sigma(1)} + \eta_1$ is an idempotent of the type described in Lemma 4.2. By Corollary 4.4, there exists $\theta_{\sigma(1)} \in \mathscr{J}$ such that

$$\mathscr{E}_{\sigma(1)} = E^{\sigma(1)} + \eta_1 = E^{\sigma(1)} + E^{\sigma(1)}\theta_{\sigma(1)} + \theta_{\sigma(1)}E^{\sigma(1)} + \theta_{\sigma(1)}E^{\sigma(1)}\theta_{\sigma(1)}.$$

Consider now the element $\mathscr{U}_{\sigma(1)}$, given by Lemma 4.5. We have

$$\left(C_{\mathscr{U}_{\sigma(1)}} \circ \varphi\right)(E^1) = C_{\mathscr{U}_{\sigma(1)}}\left(\mathscr{E}_{\sigma(1)}\right) = E^{\sigma(1)},$$

$$\overline{C_{\mathscr{U}_{\sigma(1)}} \circ \varphi} = \overline{\varphi},$$

due to Lemmas 4.5 and 4.6(i). Then

$$\left(\overline{C_{\mathscr{U}_{\sigma(1)}} \circ \varphi}\right)(E^j + \mathscr{J}) = E^{\sigma(j)} + \mathscr{J} \qquad \forall_j \in I.$$

Similarly,

$$\left(C_{\mathscr{U}_{\sigma(1)}} \circ \varphi\right)(E^2) = E^{\sigma(2)} + \eta_2, \qquad \text{where} \quad \eta_2 \in \mathscr{J}.$$

We now consider the element $\theta_{\sigma(2)} \in \mathscr{J}$, and also the idempotent $\mathscr{E}_{\sigma(2)} = E^{\sigma(2)} + \eta_2 = E^{\sigma(2)} + E^{\sigma(2)}\theta_{\sigma(2)} + \theta_{\sigma(2)}E^{\sigma(2)} + \theta_{\sigma(2)}E^{\sigma(2)}\theta_{\sigma(2)}$ and the element $\mathscr{U}_{\sigma(2)}$, given by Corollary 4.4 and Lemma 4.5 respectively. We have that

$$E^{\sigma(1)}\mathscr{E}_{\sigma(2)} = \left(C_{\mathscr{U}_{\sigma(1)}} \circ \varphi\right)(E^1E^2) = 0,$$

$$\mathscr{E}_{\sigma(2)}E^{\sigma(1)} = \left(C_{\mathscr{U}_{\sigma(1)}} \circ \varphi\right)(E^2E^1) = 0.$$

By Lemma 4.6(ii), we have that $C_{\mathscr{U}_{\sigma(2)}}(E^{\sigma(1)}) = E^{\sigma(1)}$. Therefore, $(C_{\mathscr{U}_{\sigma(2)}} \circ C_{\mathscr{U}_{\sigma(1)}} \circ \varphi)(E^1) = E^{\sigma(1)}$. Furthermore,

$$\left(C_{\mathscr{U}_{\sigma(2)}} \circ C_{\mathscr{U}_{\sigma(1)}} \circ \varphi\right)(E^2) = E^{\sigma(2)},$$

$$\overline{C_{\mathscr{U}_{\sigma(2)}} \circ C_{\mathscr{U}_{\sigma(1)}} \circ \varphi} = \overline{\varphi}.$$

That is,

$$\left(\overline{C_{\mathscr{U}_{\sigma(2)}} \circ C_{\mathscr{U}_{\sigma(1)}} \circ \varphi}\right)(E^j + \mathscr{J}) = E^{\sigma(j)} + \mathscr{J} \qquad \forall j \in I.$$

Proceeding in this way, we get that the element $\mathscr{U} = \mathscr{U}_{\sigma(n)}\mathscr{U}_{\sigma(n-1)} \cdots \mathscr{U}_{\sigma(1)}$ verifies the thesis. ■

Now we shall deal with the subgroup $\mathscr{S}$. Before stating the next lemma, we must fix some notation.

We point out that if C, C' are equivalence classes of $\bar{\rho}$ and $(x, y) \in \rho$, with $x \in C$ and $y \in C'$, then $C \times C' \subset \rho$. Among the equivalence classes of $\bar{\rho}$, we may have some classes C with the following property: if C' is an equivalence class such that either $C' \times C \subset \rho$ or $C \times C' \subset \rho$, then $C' = C$. Let $C_1, C_2, \dots, C_q$ be such classes (if they do exist), and fix an element x_i in each C_i. Choose also an element $y_l \in I$ such that y_l is a vertex of T_l for each tree T_l. In regard to the classes C_i and the trees T_l, we remark that:

(1) The sets of vertices of the trees T_l are mutually disjoint.
(2) None of the classes C_i, $1 \leqslant i \leqslant q$, intercept these sets of vertices.
(3) The union of the set of vertices of the graph $\bigcup_l T_l$ with $\bigcup_{i=1}^q C_i$ is the set I.

We are ready now to state our next lemma.

LEMMA 4.8. *Let $g : \rho \to K^*$ be a trivial function such that $g(j, k) = 1$ for all $(j, k) \in \bar{\rho}$, and let x_i, $1 \leqslant i \leqslant q$, and y_l be the elements above. Then there exists a map $s : I \to K^*$ such that*

(i) *$s(x_i) = 1$, $1 \leqslant i \leqslant q$, and $s(y_l) = 1$ for all l;*
(ii) *$g(j, k) = s(j)s(k)^{-1}$ for all $(j, k) \in \rho$.*

Proof. There exists a map $s_1 : I \to K^*$ such that

$$g(j, k) = s_1(j)s_1(k)^{-1} \quad \text{for all} \quad (j, k) \in \rho,$$

since g is trivial.

Pick $j \in I$. Due to the remarks above, we have that either $j \in C_i$ for some i which is uniquely determined, or there exists l, also uniquely determined, such that j is a vertex of T_l. In the first case, we define $s(j) = s_1(j)s_1(x_i)^{-1}$, and in the second one, we set $s(j) = s_1(j)s_1(y_l)^{-1}$. Hence, s satisfies condition (i).

To prove (ii), pick first $(j, k) \in \bar{\rho}$. Then, if $j \in C_i$ for some i, we must have that $k \in C_i$, and

$$s(j)s(k)^{-1} = s_1(j)s_1(x_i)^{-1}s_1(k)^{-1}s_1(x_i) = s_1(j)s_1(k)^{-1} = g(j,k).$$

Suppose now that $j \notin C_i$ for any i. Then, on the one hand, we have that $j, k \in C$, for some equivalence class C of $\bar{\rho}$. On the other hand, there exists an equivalence class $C' \neq C$ such that either $C \times C' \subset \rho \setminus \bar{\rho}$ or $C' \times C \subset \rho \setminus \bar{\rho}$. Then it is easy to conclude that j and k are vertices of the same tree T_l. Therefore,

$$s(j)s(k)^{-1} = s_1(j)s_1(y_l)^{-1}s_1(k)^{-1}s_1(y_l) = s_i(j)s_1(k)^{-1} = g(j,k).$$

Similarly, if $(j, k) \in \rho \setminus \bar{\rho}$, we have that j and k are vertices of the same tree T_l, and the equation above shows that $g(j, k) = s(j)s(k)^{-1}$. ∎

We remark that it can be proved that the function s in the conditions of Lemma 4.8 is uniquely determined.

We now observe that the set

$$G = \{g : \rho \to K^* \mid g \text{ is a transitive function}\}$$

is an abelian group with respect to the pointwise multiplication of functions. Furthermore, the subsets

$$D = \{g \in G \mid g \text{ is trivial}\},$$

$$F = \left\{g \in G \,\middle|\, g(j,k) = 1 \text{ for all } (j,k) \in \bar{\bar{\rho}} \cup \prod_{i=1}^{q} C_i \times C_i\right\}$$

are easily seen to be subgroups of G.

LEMMA 4.9. *With the notation above, we have that*

$$G = D \times F \qquad (\textit{direct product}).$$

Proof. First, consider $g \in D \cap F$. We claim that $g(j,k) = 1$ for all $(j,k) \in \bar{\rho}$.

In fact, given $(j,k) \in \bar{\rho}$ such that $(j,k) \notin \Pi C_i \times C_i$, there exist equivalence classes C and $C'$of $\bar{\rho}$, $C \neq C'$, satisfying $j, k \in C$ and either $C \times C' \subset \rho$ or $C' \times C \subset \rho$. Hence, j, k are vertices of the same tree T_l.

Furthermore, there exists a function $s_1 : I \to K^*$ such that $g(x,y) = s_1(x)s_1(y)^{-1}$ for all $(x,y) \in \rho$. Since $g \in F$, an easy argument shows that $s_1(j) = s_1(k) = s_1(y_l)$, where y_l is the fixed vertex of T_l. That is, $g(j,k) = s_1(j)s_1(k)^{-1} = 1$. The claim is proved.

Given the elements x_i and y_l of Lemma 4.8, we have that there exists a map $s : I \to K^*$ such that s satisfies condition (i) and $g(j,k) = s(j)s(k)^{-1}$ for all $(j,k) \in \rho$. Take $j \in I$ such that j is vertex of a (unique) tree T_l. An easy argument shows that $s(j) = 1$, since $s(y_l) = 1$ and $g(j,k) = s(j)s(k)^{-1}$ for all $(j,k) \in \bar{\bar{\rho}}$.

If $j \in C_i$ for some i then $(x_i, j) \in \bar{\rho}$. Since $g(j,k) = 1$ for all $(j,k) \in \bar{\rho}$, we have that $s(j) = s(x_i) = 1$. Hence, $g(j,k) = s(j)s(k)^{-1} = 1$ for all $(j,k) \in \rho$. Thus, $D \cap F = 1$.

Now choose an element $g \in G$, and consider the restriction of g to $\bar{\bar{\rho}}$. We claim that there exists $s : I \to K^*$ such that

$$g(j,k) = s(j)s(k)^{-1}$$

for all $(j,k) \in \bar{\bar{\rho}}$. In fact, if $j \in C_i$ for some i, we have that $(x_i, j) \in \rho$, and we define $s(j) = g(x_i, j)^{-1}$. On the other hand, if j is a vertex of a (unique) tree T_l, consider the unique path $z_0 z_1 \dots z_m$ connecting y_l to j (that is, $z_0 = y_l$ and $z_m = j$).

In order to define $s(j)$, we proceed by induction on m. If $m = 0$, then $j = y_l$ and we define $s(j) = 1$. Suppose now that $s(z_0), s(z_1), \dots, s(z_{m-1})$ are defined. As usually, either $(z_{m-1}, z_m) \in \rho$ or $(z_m, z_{m-1}) \in \rho$. In the first case, we define

$$s(j) = s(z_m) = g(z_{m-1}, z_m)^{-1} s(z_{m-1}),$$

while in the second case, we set

$$s(j) = s(z_m) = g(z_m, z_{m-1}) s(z_{m-1}).$$

Then s is as required.

In order to see this, we observe that if $(j,k) \in \bar{\bar{\rho}}$, then j and k are vertices of the same tree T_l and there exists a path joining y_l to k. Suppose that $w_0 w_1 \ldots w_t$ is such a path, with $w_0 = y_l$ and $w_t = k$. Due to the uniqueness of the paths joining vertices of T_l, we conclude that either $w_{t-1} = j$ or $w_0 w_1 \ldots w_t w_{t+1}$, with $w_{t+1} = j$, is the unique path connecting y_l to j. In both cases, the definition of s shows that $g(j, k) = s(j)s(k)^{-1}$ for all $(j, k) \in \bar{\bar{\rho}}$.

Now, let $\tilde{s}(j, k) = s(j)s(k)^{-1}$ for all $(j, k) \in \rho$. Then $\tilde{s} \in D$. Furthermore, if $(j, k) \in C_i \times C_i$ for some i, we have that

$$(\tilde{s}^{-1}g)(j, k) = g(x_i, j)g(x_i, k)^{-1}g(j, k) = g(x_i, j)g(k, x_i)g(j, k)$$
$$= g(k, j)g(j, k) = 1.$$

Therefore, we conclude that

$$(\tilde{s}^{-1}g)(j, k) = 1 \qquad \text{for all} \quad (j, k) \in \bar{\bar{\rho}} \cup \Pi C_i \times C_i.$$

Hence, $\tilde{s}^{-1}g = f \in F$, or $g = \tilde{s}f$, and the proof is complete. ∎

LEMMA 4.10. *Let $g : \rho \to K^*$ be a transitive mapping. Then $g^* = C_A$ for some $A \in S$ if and only if g is trivial.*

Proof. Suppose that $g^* = C_A$, for some $A \in S$. Then

$$AE^{ij}A^{-1} = g(i, j)E^{ij},$$

i.e.,

$$AE^{ij} = g(i, j)E^{ij}A,$$

for all $(i, j) \in \rho$. It is now easy to see that A is a diagonal matrix and hence $A_{kk} \neq 0$, for all $k \in I$. The (i, j) entries of both sides of the equation above give us $A_{ii} = g(i, j)A_{jj}$. Hence, $g(i, j) = A_{ii}A_{jj}^{-1}$, and it is enough to define $s(i) = A_{ii}$ for all $i \in I$.

Suppose now that $g(i, j) = s(i)s(j)^{-1}$ for all $(i, j) \in \rho$. Then, if $A = \Sigma_i s(i)E^i$, it is easy to see that

$$AE^{ij}A^{-1} = g^*(E^{ij}),$$

as we wished to prove. ∎

Proof of Theorem C. Let ϕ be an automorphism of S, and $\bar{\phi}$ be the automorphism induced on $\frac{S}{\mathcal{J}}$. As we saw in the proof of Proposition 4.1, we have that $\frac{S}{\mathscr{J}} \cong S'$, where

$$S' = \left\{A \in M_n(K) \middle| A_{ij} = 0 \text{ if } (i,j) \notin \bar{\rho}\right\}.$$

Furthermore, an isomorphism $f : S' \to \frac{s}{\mathscr{J}}$ is given by

$$f(A) = A + \mathscr{J} \qquad \text{for all} \quad A \in S'.$$

Due to Theorem A applied to the semisimple structural matrix algebra S' and the definition of f, we have that

$$\bar{\phi} = C_{A+\mathscr{J}} \circ \hat{\hat{\sigma}},$$

where $A \in S' \subset S$ is an invertible matrix and σ is an automorphism of $\bar{\rho}$ such that

$$\hat{\hat{\sigma}}(E^{ij} + \mathscr{J}) = E^{\sigma(i)\sigma(j)} + \mathscr{J} \qquad \text{for all} \quad (i,j) \in \bar{\rho}.$$

Furthermore, σ verifies the condition that $\sigma(x) < \sigma(y)$ whenever $(x, y) \in \bar{\rho}$ and $x < y$. Then

$$\overline{C_{A^{-1}} \circ \phi} = C_{A^{-1}+\mathscr{J}} \circ \bar{\phi} = C_{A^{-1}+\mathscr{J}} \circ C_{A+\mathscr{J}} \circ \hat{\hat{\sigma}} = \hat{\hat{\sigma}}.$$

Now, due to the definition of $\hat{\hat{\sigma}}$, we have that

$$\overline{C_{A^{-1}} \circ \phi}(E^j + \mathscr{J}) = \hat{\hat{\sigma}}(E^j + \mathscr{J}) = E^{\sigma(j)} + \mathscr{J} \qquad \text{for all} \quad j \in I.$$

Hence, the automorphism $\varphi = C_{A^{-1}} \circ \phi$ satisfies the hypothesis of Lemma 4.7. So there exists an invertible element $\mathscr{U} \in S$ such that

$$(C_{\mathscr{U}} \circ \varphi)(E^j) = E^{\sigma(j)} \qquad \text{for all} \quad j \in I,$$
$$\overline{C_{\mathscr{U}}} = 1.$$

We set $\Psi = C_{\mathscr{U}} \circ \varphi = C_{\mathscr{U}A^{-1}} \circ \phi$. Then

$$\Psi(E^{ij}) = \Psi(E^i E^{ij} E^j) = \Psi(E^i)\Psi(E^{ij})\Psi(E^j)$$
$$= E^{\sigma(i)}\Psi(E^{ij})E^{\sigma(j)} = a_{\sigma(i)}a_{\sigma(j)}E^{\sigma(i)\sigma(j)},$$

where $a_{\sigma(i)\sigma(j)} \in K^*$. Then $(\sigma(i), \sigma(j)) \in \rho$ for all $(i,j) \in \rho$; hence σ is actually an automorphism of ρ, and $\hat{\sigma} \in \mathscr{P}$.

Now, let $g : \rho \to K^*$ be the transitive function

$$g(\sigma(i), \sigma(j)) = a_{\sigma(i)\sigma(j)} \qquad \text{for all} \quad (i,j) \in \rho.$$

We have that

$$\Psi(E^{ij}) = g^*(E^{\sigma(i)\sigma(j)}),$$

i.e.,

$$\Psi(E^{ij}) = (g^* \circ \hat{\sigma})(E^{ij}),$$

for all $(i,j) \in \rho$. Hence $\Psi = g^* \circ \hat{\sigma}$, where $\hat{\sigma} \in \mathscr{P}$, so

$$C_{\mathscr{U}A^{-1}} \circ \phi = g^* \circ \hat{\sigma},$$

i.e.,

$$\phi = C_{A\mathscr{U}^{-1}} \circ g^* \circ \hat{\sigma}.$$

But, by Lemma 4.9, we have that $g = df$, where $d \in D$ and $f \in F$. Hence, $g^* = d^* \circ f^*$, where $f^* \in \mathscr{G}$ and $d^* = C_{A'}$ for some $A' \in S$, due to Lemma (4.10). Then

$$\phi = C_{A\mathscr{U}^{-1}A'} \circ f^* \circ \hat{\sigma},$$

where $C_{A\mathscr{U}^{-1}A'} \in \mathscr{C}$, $f^* \in \mathscr{G}$, and $\hat{\sigma} \in \mathscr{P}$. Thus, ϕ has the required form.

In order to finish the proof, we must show that:

(a) $\mathscr{C} \cap \mathscr{G} = 1$, $g^* \circ C_A \circ (g^*)^{-1} \in \mathscr{C}$ for all $g^* \in \mathscr{G}$ and $C_A \in \mathscr{C}$.

(b) $(\mathscr{C} \rtimes \mathscr{G}) \cap \mathscr{P} = 1$, $\hat{\sigma} \circ \phi \circ (\hat{\sigma})^{-1} \in \mathscr{C} \rtimes \mathscr{G}$ for all $\hat{\sigma} \in \mathscr{P}$ and $\phi \in \mathscr{C} \rtimes \mathscr{G}$.

We start with claim (a). Clearly, $\mathscr{C}$ is normal in Aut S. So pick $g^* \in \mathscr{G}$ such that g^* is conjugation by a matrix of S. By Lemma 4.10, g is trivial. Hence, $g \in D \cap F$. By Lemma 4.9, we obtain $g = 1$, i.e., $g^* = 1$.

To prove (b), we observe that it is enough to show that $\hat{\sigma} \circ g^* \circ (\hat{\sigma})^{-1} \in \mathscr{C} \rtimes \mathscr{G}$ for all $g^* \in \mathscr{G}$. We have that

$$(\hat{\sigma} \circ g^*)(E^{jk}) = \hat{\sigma}(g(j,k)E^{jk}) = g(j,k)E^{\sigma(j)\sigma(k)} \qquad \text{for all} \quad (j,k) \in \rho.$$

Let $h : \rho \to K^*$ be such that $h(\sigma(j), \sigma(k)) = g(j, k)$ for all $(j, k) \in \rho$. Trivially, h is a transitive function. Therefore, $h \in \mathscr{G}$. By Lemma 4.9, $h = df$, where $d \in D$ and $f \in F$. Applying this to the equation above, we obtain

$$(\hat{\sigma} \circ g^*)(E^{jk}) = (h^* \circ \hat{\sigma})(E^{jk}) = (d^* \circ f^* \circ \hat{\sigma})(E^{jk})$$

$$\text{for all} \quad (j, k) \in \rho.$$

That is,

$$\hat{\sigma} \circ g^* \circ (\hat{\sigma})^{-1} = d^* \circ f^*.$$

But, by Lemma (4.10), we have that $d^* \in \mathscr{C}$. Therefore, $d^* \circ f^* \in \mathscr{C} \rtimes \mathscr{G}$, as we wanted to prove.

Finally, pick $\hat{\sigma} \in (\mathscr{C} \rtimes \mathscr{G}) \cap \mathscr{P}$. We have that $\hat{\sigma} = C_A \circ g^*$, where $C_A \in \mathscr{C}$ and $g^* \in \mathscr{G}$. Then

$$E^{\sigma(j)} = \hat{\sigma}(E^j) = (C_A \circ g^*)(E^j) = C_A(E^j) \qquad \text{for all} \quad j \in I.$$

Considering this equation in $\frac{S}{\mathscr{J}}$, we get that

$$E^{\sigma(j)} + \mathscr{J} = \mathscr{C}_{A+\mathscr{J}}(E^j + \mathscr{J}).$$

Now, we recall that $\frac{S}{\mathscr{J}} \cong S'$, and from the definition of the isomorphism, we get that

$$E^{\sigma(j)} = C_{A'}(E^j),$$

where $A' \in S'$ is such that $A' + \mathscr{J} = A + \mathscr{J}$. But conjugation by A' fixes the simple components of S'. By Lemma 3.6, we conclude that $\sigma(j)$ belongs to the equivalence class of j defined by the relation $\bar{\rho}$. From the definition of $\mathscr{P}$, we get that $\sigma = 1$.

The proof of Theorem C is complete. ∎

Proof of Theorem D

Suppose first that conditions (i) and (ii) hold. Then Lemmas 4.9 and 4.10 and the definition of $\mathscr{P}$ imply respectively that $\mathscr{G} = 1$ and $\mathscr{P} = 1$. Hence, by Theorem C, every automorphism of S is inner.

Now, assume that every automorphism of S is inner, and consider a transitive function $g : \rho \to K^*$. By hypothesis g^* is inner. Hence, by Lemma 4.10, g is trivial.

For (ii), pick an automorphism σ of ρ. Since $\hat{\sigma}$ is inner, computing in $\frac{S}{\mathscr{J}}$, we have

$$E^{\sigma(j)} + \mathscr{J} = \bar{\hat{\sigma}}(E^j + \mathscr{J}) = C_{A+\mathscr{J}}(E^j + \mathscr{J}).$$

Now, $\frac{S}{\mathscr{J}} \cong S'$, where

$$S' = \left\{A \in M_n(K) \middle| A_{ij} = 0 \text{ if } (i,j) \notin \vec{\rho}\right\},$$

as we saw in Proposition 4.1. So $E^{\sigma(j)} = C_{A'}(E^j)$, where $A' \in S$ is such that $A' + \mathscr{J} = A + \mathscr{J}$.

Since S' is semisimple and $C_{A'}$ fixes the simple components of S', we get from Lemma (3.6) that $\sigma(j)$ belongs to the equivalence class of j defined by the relation $\bar{\rho}$. This completes the proof. ■

We can now obtain the results from [3] and [6] in the case where the ring of coefficients is a field.

COROLLARY 4.11. *Every K-automorphism of the algebra of upper triangular matrices with entries on K is inner.*

Proof. We observe first that this is a structural matrix algebra defined by the relation

$$\rho = \{(i,j) | i,j \in I \text{ and } i \leqslant j\}.$$

Hence, we must show that conditions (i) and (ii) of Theorem D hold.

To prove (i), pick a transitive function $g : \rho \to K^*$. We define

$$s(1) = 1, \qquad s(2) = g(1,2)^{-1}, \ldots, \qquad s(n) = g(1,n)^{-1}.$$

We have that $g(1,i)g(i,j) = g(1,j)$, that is, $g(i,j) = s(i)s(j)^{-1}$. Hence, g is trivial.

For (ii), pick an automorphism σ of ρ. Since the equivalence classes of $\bar{\rho}$ are singleton, we must show that $\sigma = 1$.

We observe first that the relation ρ can be written as $\rho = \bigcup_{i=1}^{n} \rho_i$, where $\rho_i = \{(i,i), (i, i+1), \ldots, (i,n)\}$, $1 \leqslant i \leqslant n$. Then the image of ρ_1 by

σ is $\{(\sigma(1), \sigma(1)), (\sigma(1), \sigma(2)), \ldots, (\sigma(1), \sigma(n))\}$. Now, this set has n elements and is contained in ρ_i for some i. But $|\rho_i| = n - i + 1$, so $i = 1$. Therefore, $\sigma(1) = 1$.

Proceeding in the same way with ρ_2, we get that $\sigma(2) = 2$ and, similarly, $\sigma(i) = i$ for all i. Hence, $\sigma = 1$. ∎

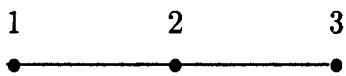
5. EXAMPLES

We now show that conditions (i) and (ii) of Theorem D are mutually independent.

EXAMPLE 1. Let K be an arbitrary field. We exhibit a relation ρ where every transitive mapping $g : \rho \to K^*$ is trivial, but not all automorphisms of ρ fix the equivalence classes of $\bar{\rho}$.

Set $\rho = \{(1,1), (1,2), (2,2), (3,2), (3,3)\}$; so $\bar{\rho} = \{(1,1), (2,2), (3,3)\}$. Now, the permutation $\sigma = (1\,3)$ is easily seen to be an automorphism of ρ, but $\sigma(1) \neq 1$.

Let now $g : \rho \to K^*$ be a transitive function. We observe that the graph Δ associated to $p \setminus \bar{\rho}$ is



Hence, it is connected and coincides with a tree containing all its vertices. Therefore, with the notation of Lemma 4.9, we have that $G = D$, $F = 1$. That is, every transitive function is trivial.

We point out that, for the corresponding structural matrix algebra, we have that $\mathcal{G} = 1$ and $\mathcal{P} = \{\hat{1}, \hat{\sigma}\}$.

EXAMPLE 2. We shall now show that we can find a field K and relation ρ where the automorphisms fix the equivalence classes of $\bar{\rho}$, but not all transitive functions $g : \rho \to K^*$ are trivial.

Set

$$\rho = \{(1,1), (1,4), (1,5), (1,6), (2,2), (2,3), (2,4), (2,5), (2,6),$$
$$(3,2), (3,3), (3,4), (3,5), (3,6), (4,4), (5,5), (5,6), (6,5), (6,6)\},$$

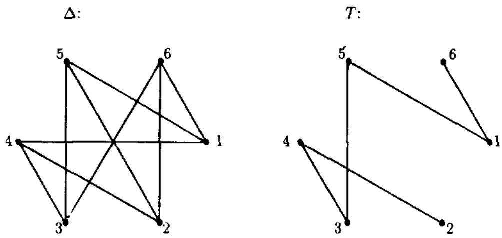


FIG. 1.

and let K be a field such that $|K| > 2$. For such a ρ, we have that the graph Δ and a tree T which contains all its vertices are given by Figure 1. Furthermore, we have that $\bar{\bar{\rho}} = \{(1,5),(1,6),(2,4),(3,4),(3,5)\}$ and $\bar{\bar{\bar{\rho}}} = \varnothing$. Then the function $g : \rho \to K^*$ such that $g(1,4) = \alpha \in K^*$, $\alpha \neq 1$, $g(i,j) = 1$ for all $(i,j) \neq (1,4)$ is easily seen to be a nontrivial transitive function.

We now observe that the equivalence classes of $\bar{\rho}$ are $\{1\},\{2,3\},\{4\},\{5,6\}$. Let σ be an automorphism of ρ. Since equivalence classes go to equivalence classes under σ, we have that either $\sigma(1) = 1$ or $\sigma(1) = 4$. But if $\sigma(1) = 4$, we must have that $\sigma(4) = 1$, and then $(\sigma(1), \sigma(4)) = (4,1)$, which does not belong to ρ. Then $\sigma(1) = 1$ and therefore $\sigma(4) = 4$.

Similarly, either $\sigma(\{2,3\}) = \{2,3\}$ or $\sigma(\{2,3\}) = \{5,6\}$. If $\sigma(2) = 5$, we have that $(\sigma(2), \sigma(4)) = (5,4) \notin \rho$, a contradiction. Hence, $\sigma(\{2,3\}) = \{2,3\}$ and consequently $\sigma(\{5,6\}) = \{5,6\}$.

An easy calculation shows that for the corresponding structural matrix algebra, we have that $\mathcal{G} \cong K^*$ and $\mathcal{P} = 1$. ∎

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